General Relativity $\boxed{\text{Week 5}}$

<u>Normal coordinates around $p \in M$</u> (in a normal neighborhood)

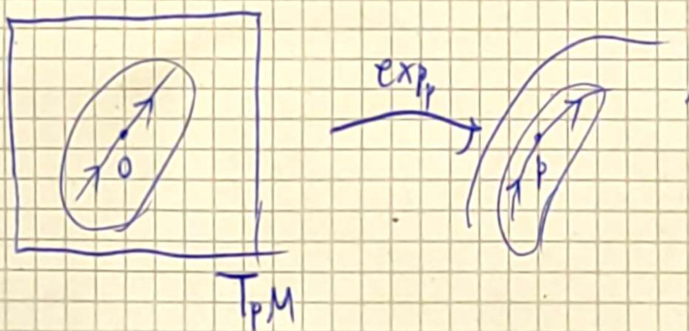
Choose orthonormal frame $\{e_0, \dots, e_n\}$ of $(T_pM, g|_p)$

So for $v \in T_pM$: $v = v^a e_a$

Define coordinates $(x^0, x^1, \dots, x^n)$ on M around p:

$$x^a(\exp_p(v)) = v^a \iff (\exp_p^{-1}(q))^a = x^a(q) \quad \text{for } q \text{ near } p.$$

In these coordinates:



$\gamma_{p,v}(t) = \exp_p(vt)$

becomes $\gamma^k_{p,v}(t) = v^k t$

<u>Prop:</u> If $(x^0, \dots, x^n)$ are normal coordinates around $p \in (M, g)$, then

$$g_{\alpha\beta}(0) = \eta_{\alpha\beta}, \qquad \partial_\alpha g_{\beta\gamma}(0) = 0, \qquad \Gamma^\alpha_{\beta\gamma}(0) = 0$$

<u>Proof:</u> Note that the x^a coordinate curve through p is $\exp_p(e_a t)$

$\frac{\partial}{\partial x^a}$: the tangent vector to the x^a coordinate curve. So

$$\left.\frac{\partial}{\partial x^a}\right|_p = \frac{d}{dt}\left(\exp_p(e_a t)\right)\Big|_{t=0} = e_a, \quad \text{so}$$

$$g_{\alpha\beta}(0) = g\left(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta}\right)\Big|_p = g(e_\alpha, e_\beta) = \eta_{\alpha\beta} \quad \text{(orthonormal basis).}$$

And: For any $v \in T_pM$:

$\gamma^k_{p,v}(t) = v^k t$ is a geodesic, so $\ddot{\gamma}^k_{p,v} + \Gamma^k_{\alpha\beta}(\gamma(t)) \cdot \dot{\gamma}^\alpha \dot{\gamma}^\beta = 0$

$$\Rightarrow \Gamma^k_{\alpha\beta}(\gamma(t))\, v^\alpha v^\beta = 0 \overset{t=0}{\Longrightarrow} \Gamma^k_{\alpha\beta}(0)\, v^\alpha v^\beta = 0 \qquad \forall v \in T_pM$$

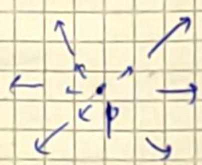
$$\xRightarrow[\text{in } \alpha,\beta]{\Gamma^k_{\alpha\beta} \text{ symmetric}} \Gamma^k_{\alpha\beta}(0) = 0.$$

And $\partial_\alpha g_{\beta\gamma} = g_{\beta\lambda}\Gamma^\lambda_{\alpha\gamma} + g_{\gamma\lambda}\Gamma^\lambda_{\alpha\beta} \Rightarrow \partial_\alpha g_{\beta\gamma}(0) = 0.$ ☐

Moreover, the Gauss-Lemma holds:
(with the same proof as in the Riemannian case, albeit using the fact that geodesics are stationary "points" of the energy functional, rather than the length functional):

In normal coordinates around p, at every point $q = (x^0, \dots, x^n)$:

$$\boxed{g_{\alpha\beta}\, x^\alpha = \eta_{\alpha\beta}\, x^\alpha}$$



$S^\alpha = x^\alpha$ is the radial vector field

Differentiating 3-times and evaluating at $x=0$:

$$\boxed{\partial^2_{\alpha\beta}\, g_{\gamma\delta}(0) + \partial^2_{\gamma\alpha}\, g_{\beta\delta}(0) + \partial^2_{\beta\gamma}\, g_{\alpha\delta}(0) = 0.}$$

Local causal structure:

Recall: If (M,g) is a spacetime,

$$I^+_{(M,g)}[p] = \{ q \in M : \exists \text{ fut. directed timelike curve } p \to q \} \subseteq M$$

While $I^+_p = \{ v \in T_pM : v \text{ fut. directed timelike} \} \subset T_pM$

Locally: The causal structure around p is similar to that around a point in Minkowski spacetime (i.e. locally causality is as in special relativity)

More precisely:

Prop: Let (M,g) be a spacetime and U_p be a normal neighborhood of p, such that $\exp_p : V_p \longrightarrow U_p$, $V_p \subset T_pM$, is a diffeomorphism. Then:

$$I^+_{(U_p,g)}[p] = \exp_p\left(V_p \cap I^+_p\right).$$

In particular, if $q \in I^+_{U_p}[p]$, then $\exists$ timelike geodesic from p to q.

Remarks: • In general, $I^+_{U_p}[p] \subseteq I^+_M[p] \cap U_p$, but might not be equal

Ex:

$\leftarrow M: S^1 \times \mathbb{R}$: $I^+_M[p] = M$.

• From the proof of the proposition:

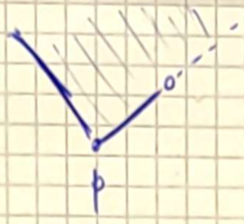
$$J^+_{U_p}[p] \setminus I^+_{U_p}[p] = \exp_p\left(V_p \cap (J^+_p \setminus I^+_p)\right), \text{ so}$$

$J^+_{U_p}[p] \setminus I^+_{U_p}[p]$ is a hypersurface foliated by null geodesics emanating from p

• $\overline{I^+_{U_p}[p]} = J^+_{U_p}[p]$

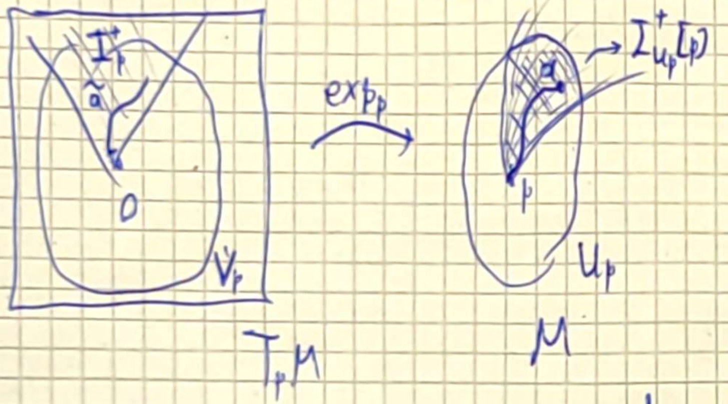
In general: We have $\supseteq$, but not necessarily $=$, e.g.:

$(\mathbb{R}^{1+1}, \eta)$ minus a point:



Proof: One direction is simple: $\exp_p(I_p^+ \cap V_p) \subseteq I^+_{U_p}[p]$,

since the first set contains points connected to p with a timelike geodesic segment.



For the opposite direction:
Let $q \in I^+_{U_p}[p]$. This means that there exists a timelike, future directed curve $\alpha : [0,1] \to U_p$ such that $\alpha(0) = p$, $\alpha(1) = q$. We need to show that $\tilde{\alpha}(t) = \exp_p^{-1}(\alpha(t))$ stays inside I_p^+.

Denoting with $(x^0, \dots, x^n)$ the normal coordinates on U_p: Define

$$F : U_p \to \mathbb{R}, \quad F(x) = -(x^0)^2 + (x^1)^2 + \dots (x^n)^2$$

and $\tilde{F} : V_p \to \mathbb{R}$, $\tilde{F}(v) = F(\exp_p(v))$. Note: $\tilde{F}$ is the "Lorentzian norm".

$$\tilde{\alpha} \text{ staying inside } I_p \iff \tilde{F}(\tilde{\alpha}(t)) < 0 \text{ for } t \in [0,1]$$

$$\iff \underbrace{F(\alpha(t))}_{F(t)} < 0 \text{ for } t \in [0,1]$$

(but I_p has two components; if $\tilde{\alpha}(t_0) \in I_p^+$, then it will remain in that one)

If $\alpha(t) = (x^0(t), \dots, x^n(t))$:

$$F(\alpha(t)) = \eta_{\alpha\beta}\, x^\alpha(t)\, x^\beta(t), \quad \dot{F}(\alpha(t)) = 2\eta_{\alpha\beta}\, x^\alpha(t) \cdot \dot{x}^\beta(t),$$

$$\ddot{F}(\alpha(t)) = 2\eta_{\alpha\beta}\, \dot{x}^\alpha(t) \cdot \dot{x}^\beta(t) + 2\eta_{\alpha\beta}\, x^\alpha(t)\, \ddot{x}^\beta(t)$$

• At $t=0$: $F(0)=0$, $\dot F(0)=0$, $\ddot F(0)<0$ since $\dot a(0)$ is timelike ($\eta_{\mu\nu}=g_{\mu\nu}(0)$)

So $F(t)<0$ for $t\in(0,\delta)$

• By continuity: It suffices to show that for any $t\in(0,1)$, if $F(t)<0$ then $\dot F(t)\leq 0$ (since this means that we cannot reach a point with $F(t)\geq 0$)

By Gauss-Lemma:

Radial vector

$$\dot F(t) = 2\,\eta_{\alpha\beta}\,x^{\alpha}(t)\,\dot x^{\beta}(t) = 2\,g_{\alpha\beta}(x(t))\,x^{\alpha}(t)\,\dot x^{\beta}(t)$$

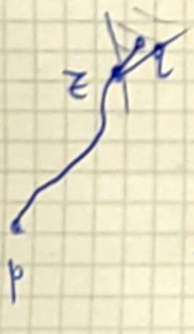
$$= 2\,g\left(\dot\gamma_{p,\exp_p^{-1}(a)}(t),\ \dot a(t)\right)$$

Timelike vectors (since $\exp_p^{-1}(a)$ is timelike if $F(a)<0$)

So inner product <0. ☐

<u>Corollary</u>: If $S\subseteq M$, $I^+(S)$ is open.

<u>Proof</u>: $I^+(S)=\bigcup_{p\in S} I^+(p)$, so we need to show that $I^+(p)$ is open. Indeed: If $q\in I^+(p)$, $\exists$ timelike, future directed curve γ from p to q. Let z be a point on γ between p and q, but sufficiently close to q so that q lies in a normal neighborhood U_z of z.



Then: $\gamma|_{[z,q]}\subset U_z \Rightarrow q\in I_{U_z}[z]$

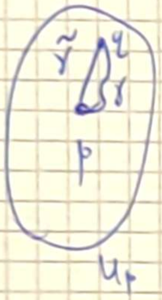
But $I_{U_z}[z]$ is open (by the previous prop) and

$I^+_{U_z}[z] \subseteq I^+[z] \subseteq I^+[p]$ (since $z \in I^+[p]$)

So $I^+[p]$ contains an open neighborhood of q $\Rightarrow$ $I^+[p]$ is open

$\boxtimes$

<ins>Local version of the twin paradox:</ins>



Let U_p be a normal neighborhood of $p \in (M,g)$ and let $q \in I^+_{U_p}[p]$. Let $\tilde{\gamma}$ denote the radial timelike geodesic from p to q. Then, for any other timelike curve γ from p_0 to q, we have

$$l(\tilde{\gamma}) \geq l(\gamma),$$

with equality iff $\tilde{\gamma} = \gamma$ up to reparametrization.

<ins>Proof:</ins>

We have $\gamma(t) = \exp_p(\sigma(t)) = \exp_p(r(t) \cdot \hat{n}(t))$,

where $r(t) = -|\sigma(t)|$, $\hat{n}(t) = \frac{\sigma(t)}{-|\sigma(t)|}$

(Note that $\sigma(t) \in I^+_p$): So $\langle \hat{n}, \hat{n} \rangle = -1$

and $\langle \hat{n}, \frac{d}{dt}\hat{n} \rangle = 0$,

therefore: $\langle \frac{d\hat{n}}{dt}, \frac{d\hat{n}}{dt} \rangle \geq 0$. *

We have $l(\gamma) = \int_0^1 \sqrt{-g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))}\, dt$ In normal coordinates: $\gamma^a(t) = r(t) \cdot \hat{n}^a(t)$

$$= \int_0^1 \sqrt{-g_{ab} \cdot (\dot{r} \cdot \hat{n}^a + r \cdot \dot{\hat{n}}^a)(\dot{r}\hat{n}^b + r \cdot \dot{\hat{n}}^b)}\, dt$$

$$= \int_0^1 \sqrt{-\dot{r}^2 \underbrace{g(\hat{n},\hat{n})}_{=-1} - 2r\dot{r}\underbrace{g(\hat{n},\dot{\hat{n}})}_{=0} + r^2 \underbrace{g(\dot{\hat{n}},\dot{\hat{n}})}_{\geq 0}}$$

$$\leq r(1) = l(\tilde{\gamma})$$

Why is $*$ true?
because $\hat{n}$ timelike, $\frac{d}{dt}\hat{n} \perp \hat{n}$ $\Big\} \Rightarrow \frac{d}{dt}\hat{n}$ is space like. ☒

Another corollary of the local theory.

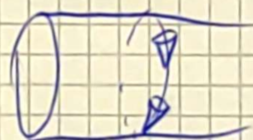
If $q \in J^+[p] / I^+[p]$: p is connected to q with a fut. directed null geodesic.

<u>Hierarchy of causality conditions</u> ← Global conditions!

Let (M, g) be a spacetime (i.e. time oriented).

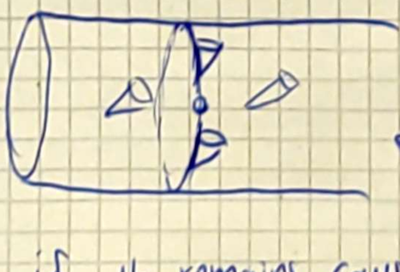
<u>Def</u>: (M, g) is called:

- <u>Causal</u>, if no closed causal curves



↑ very unstable condition

- <u>Strongly causal</u> if no "almost" closed causal curves
  i.e. $\forall p \in M$: $\exists$ open neighborhood O of p such that no causal curve intersects O more than once.



← Causal, but not strongly causal.

- <u>Stably causal</u>: if it remains causal even under small perturbations of the metric (i.e. of the time cones).

Equivalent to being stably causal (which we will adopt as an equivalent definition):

$\exists$ C^∞ function $f: M \to \mathbb{R}$ such that

$\nabla^a f = g^{a\beta} \nabla_\beta f$ is <u>timelike</u> everywhere and fut. directed

(f: <u>time function</u>).

<u>Note</u>: $\nabla^a f$ being timelike is a stable condition (since a timelike vector remains timelike under small perturbations of the metric.

If γ is a timelike and future directed curve:

$\dot{\gamma}(f) = \langle \dot{\gamma}, \nabla f \rangle < 0$ so f is

strictly decreasing along f.

The strongest causality condition:

<u>Def</u>: A <u>spacetime</u> (M,g) is <u>globally hyperbolic</u> if it admits a <u>Cauchy hypersurface</u>.

<u>Def</u>: Σ is a Cauchy hypersurface of (M,g) if every <u>inextendible</u> <u>causal curve</u> intersects Σ <u>exactly once</u>.

So Σ: cannot have timelike or null pieces.

E.g: In <u>Minkowski</u>: $(\mathbb{R}^{n+1}, \eta)$:

- $\{t = \text{const}\}$ is a Cauchy hypersurface

- $-t^2 + \sum_{i=1}^{n} (x^i)^2 = -1$ is <u>not</u>: